Understanding of Transitivity and Symmetry of Equality among High School Students

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Abstract

Effective algebraic thinking relies on a relational understanding of the equality sign, that is, that its purpose is to indicate that the values represented by the expressions on either side are the same. Students used to arithmetic thinking tend to see the equality sign operationally, that is, that its purpose is to indicate that operations on the left are to be performed and the result placed to the right. This study examines two foundational concepts in equality, transitivity and symmetry, which rely on relational rather than operational understanding. The sample set is twenty-nine high school girls at an alternative school in Metropolitan Detroit.

Submitted in partial fulfillment of the requirement for

the Masters of Arts in Teaching

with a concentration in

Secondary Mathematics Education

Wayne State University

2013

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Rationale and Literature Review

Introduction

Common Core and the National Council of Teachers of Mathematics emphasize teaching algebra at a younger age than traditional models (Falkner, Levi, & Carpenter, 1999). A relational understanding of the equality sign is key to being prepared for algebraic thinking (Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011; McNeil et al., 2006): The use of the equality sign in this manner indicates that the values of the expressions to either side of the sign are the same. However, learners often struggle with understanding the equality sign in terms of actual equivalence (Kieran, 1981); instead, students tend to see the sign operationally, to indicate that an operation is to be performed on the values to the left, with the result being placed to the right (Baroody & Ginsburg, 1983; Wheeler, 2010).

In all, Molina et. al. lists about a dozen uses of the equality sign in mathematical contexts (Molina, Castro, & Castro, 2009, 346-348). These include "proposal of an activity," "operator," "splitter," "expression of an equivalence," "expression of a conditioned equivalence," "definition of a mathematical object," and "assignment of numeric value" (among others). However, most of these can be placed into the two categories of "operational" and "relational." For instance, "splitter" involves using the equality sign to mark off steps of a problem: $3x - 4 = 5 = 3x = 9 = x = 3$. Note that every other equality sign is being used in a non-traditional way; the standard convention is to use the "implies" character instead: $3x - 4 = 5 \implies 3x = 9 \implies x = 3$. Non-standard usages such as this reinforce the operational rather than relational aspect of the equality sign.

Furthermore, this bipartite understanding of the equality sign is not limited to the United States (Molina's research is of Spanish students) nor even to our era; a half-millennium ago, before the sign itself was conventionalized, sources referred variously to "aequales" ("having the same value") or to "faciunt" ("makes") (Oksuz, n.d.).

Operational vs Relational

Rittle-Johnson et. al. (2011, 87) divide the two categories into a total of four levels:

Level 4 - Comparative relational

Learners reveal a relational understanding by successfully modifying each side of an equation in the same way (e.g., $2x + 5 = 9 \implies 2x + 5 - 5 = 9 - 5 \implies 2x = 4$).

Level 3 - Basic relational

Equations can involve operations on both sides (e.g., $4 + 2 = 5 + 1$), but the learner can't explicitly articulate the relational aspect of the equality sign.

Level 2 - Flexible operational

Equations that involve operations are acceptable even if operations appear on the right (e.g., $3 = 2 + 1$).

Level 1 - Rigid operational

Only equations that involve an operation-based expression on the left and a constant or "result" on the right (e.g., $3 + 4 = 7$) are considered valid.

They state that these levels are in order of sophistication, from greatest to least. However, this should not necessarily be taken to mean that operational understanding consistently occurs before relational understanding; Baroody and Ginsberg (1983) examined two conflicting arguments: That early elementary students struggle with the relational view because they're cognitively incapable of doing so (Kieran, 1981), and that they struggle because so much of elementary pedagogy focuses (not necessarily deliberately) on the operational view, by providing myriad worksheets of the format $3 + 4 = \square$. By looking at students that had been exposed to a specific curriculum, Wynroth Math, which de-emphasizes the standard operational-type exercises, Baroody and Ginsberg concluded that young learners are capable of understanding the equality sign relationally. Wheeler argues, however, that Baroody and Ginsberg's research isn't completely definitive, but rather does suggest *some* cognitive obstacles to relational thinking for children, just not as strong as Kieran's position (Wheeler, 2010, 12).

Falkner et. al. present data that further supports Baroody and Ginsberg's conclusions, and goes one step further: Student understanding of the relational nature of equality may actually *weaken* during the later elementary years. They surveyed students in grades one to six with questions of this nature:

$$8 + 4 = \square + 5$$

While only a few of the second, third, and fourth graders correctly answered this question (8%), none of the 145 sixth graders did so. In fact, 84% of the sixth graders apparently ignored the part after the box and gave an answer of 12, while the younger respondents gave a variety of answers, including 12 (57%) but also including 17 (16%), "12 and 17" (7%) and "other" (11%) (Falkner et al., 1999, 233). Granted, the sample sizes are low enough that the differences could be within statistical variation, but the data at least suggests that students enter middle school with a greater confidence in the wrong answer than younger students have.

Wheeler further found that, while college-age students can consistently accurately answer questions of the sort $8+4 = \square+5$, they still struggle with articulating the relational nature of the equality sign, and are prone to fall back to an operational understanding when it presents itself:

> While most college students have the ability to transition from a procedural understanding of equations to a structural understanding, they do not bring that knowledge to bear without prompting or encouragement. In a context where a procedural interpretation of an equation is consistent with a student's perception of how they should interact with the equation, they will most likely fail to interpret the equals relation. (Wheeler, 2010, 51)

Symmetry and Transitivity

Most of the research has focused on questions of the sort just described: $8+4 = \square+5$. A poor understanding of the relational nature of equality will generally lead to one of two answers: 12, by virtue of simply ignoring the material after the square (perhaps on the assumption that it will be used later; see the discussion of splitters above), or 17, by virtue of tossing all the available numbers and operators together. Four-fifths of the students in Falkner's study provided one of those two answers (Falkner et al., 1999, 233).

I was curious about two even more rudimentary concepts: Transitivity $((a = b) \wedge (b = c) \implies (a = c))$ and symmetry $((a = b) \implies (b = a))$. Common Core calls for an understanding of these concepts in First Grade (Common Core State Standards Initiative, n.d., 13). A solely operational understanding of mathematics would preclude an understanding of these concepts; indeed, research has shown that students will reject reflexive statements ($a = a$) (Molina et al., 2009, 349), the most basic and trivial of relational statements, as invalid because there's nothing at all to operate on.

In order to determine whether the research observations on fill-in-the-blank problems (like $8+4 = \square+5$) would hold up for symmetry and transitivity as well, I designed a survey instrument set to compare student responses to both the widely-tested frame above and transitive and symmetrical problems.

I was also curious about where the student disconnect comes from: Is it due to the use of mathematical equations, or is it a cognitive block for the type of underlying problem being posed? To that end, my primary instrument includes problems which are at their core about transitivity and symmetry, but which are stated in words rather than in the form of mathematical expressions. I included two questions which asked precisely the same question but in different forms so that I could compare student response differences.

Hence, this study seeks to examine two questions regarding the relational view of equality:

1. Do difficulties with the relational view extend to transitivity and symmetry?
2. Do difficulties with the relational view extend to non-mathematical contexts?

Methodology

Respondent set

Data was collected from a residential facility for adjudicated minor girls in Metropolitan Detroit. I selected two of my own classes for the study. Prior to data collection, students were informed that this was not a graded activity, and that their participation or lack thereof would not be counted towards their grade, but that I would appreciate any effort they put into the activity.

Group A was a general mathematics course. Students ranged in age from 14 to 17, with four white students and twelve black students. Two students declined to participate in the questionnaire; as a result, the sample size is fourteen. One of the students refusing the questionnaire wanted to be included in the one-on-one interviews, though.

Group B was an advisory course. Students ranged in age from 13 to 17, with one white student and twelve black students. One student declined to participate; as a result, the sample size is twelve.

The age and race breakdowns of the twenty-six students completing the questionnaire are shown in table 1.

	White	Black	Total
13	0	4	4
14	0	3	3
15	1	5	6
16	3	5	8
17	1	4	5
Total	5	21	26

Table 1: Respondent age and race

For this report, all respondents have been assigned pseudonyms. Names beginning with letters from A (Abigail) to N (Nikki) represent students from Group A; names beginning with letters from O (Olivia) to Z (Zinnia) represent students from Group B. Students were informed verbally and on the instrument that analysis of their responses would be anonymous. See Appendix D for the complete list of respondents.

Instrument

The instrument used for this study (Appendix A, Appendix B) consists of three parts:

Preview Reflection

All respondents were given a worksheet which asked them to reflect on their current knowledge about the equality sign, using two questions. These questions were intended to tease out whether students had a primarily operational or primarily relational perspective of the equality sign. Students were also asked to indicate their familiarity with four key terms: *transitive, symmetry, commutative,* and *associative*.

Numbers Worksheet

Once the respondents completed their Preview Reflections, they turned them in and were given a worksheet entitled "The Nature of Numbers." This worksheet consists of a total of fifteen questions, grouped as follows:

1. Four questions on basic mathematics, of the sort that students are used to encountering, including one Level 1 question, one Level 2 question, and two Level 3 questions (one on symmetry).
2. Six mathematical statements which students were asked to identify as true or

false. Ozsuz recommends using these sorts of questions to move away from a strictly operational perspective, since students aren't being asked to actually complete a problem (Oksuz, n.d., 10). Three of these questions concern symmetry and three concern transitivity.

3. Two plain text statements with which students are asked to agree or disagree. One of these (NN-12) is mathematically equivalent to one of the previous set (NN-10); if respondents are parsing mathematical and non-mathematical statements comparatively, they should have similar success rate with these two questions.

4. Three plain text statements in which students are asked to fill in a blank. The first two of these (NN-13 and NN-14) ask about a transitive (sibling) and non-transitive (parent) familial relationship; the last is a question about symmetry.

Group Interview

Two days after taking the survey, students from Group A were asked to be part of interview groups of two or three students each. Students were selected based on general skill level and willingness to participate. There were three such groups conducted, for a total of eight students. Unfortunately, class scheduling conflicts precluded me from conducting similar interviews with Group B respondents. Due to the request of the principal of the school, no recordings were taking; analysis is based on hand-written notes.

See Appendix C for more information on the survey design and motivation.

Results

Preview Reflection

I expected respondents to struggle with articulating responses to open-ended questions. However, overall, students provided responses that provided some detail, enough to get an understanding of general patterns of thought.

Responses to PR-1 ("Describe how you would think about…") fell into three basic categories: Real world examples (Daisy: "You add the 3 to the 5 and you get eight. Like lets say you have 3 pickles and you get five more you have eight."), tally mark or counting references (Isabella: "I would just think of my 10 fingers being up then just put 5 up on one hand and 3 on the other which = 8"), and purely mathematical, such as repeating the problem (Mimi: "I would describe the following problem by saying add 5 plus 3."). Additional examples of each are shown in figures 1 to 3.

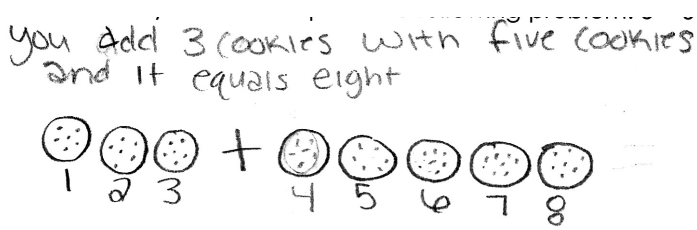

Figure 1: Real world example (Nikki)

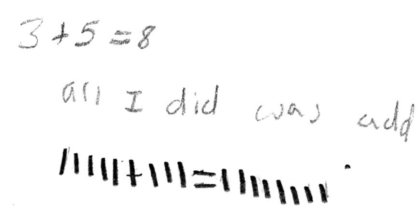

Figure 2: Tally marks (Rachel)

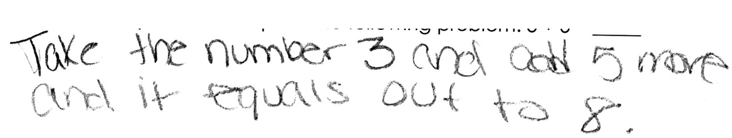

Figure 3: Mathematical explanation (Ebony)

Each type of explanation was represented equally in the data: Eight respondents provided a real world example, nine referenced tally marks or counting of some sort, and nine gave a purely mathematical explanation. All of the explanations were operational, although Quinn's did hint at playing with the notion of equivalence relationally: "$3+5=8; 5+3=8; 1+2+5=8; 3+2+3=8.$"

Responses to PR-2 ("Think about the equal sign...") were predominantly rigidly operational (Abigail: "The equal sign is a sign that will indicate the total number of the equation before it."). While this might have been primed by PR-1, most of the responses failed to directly reference PR-1, and most made it clear that the answer wasn't limited to addition, such as Kimberly's response in figure 4.

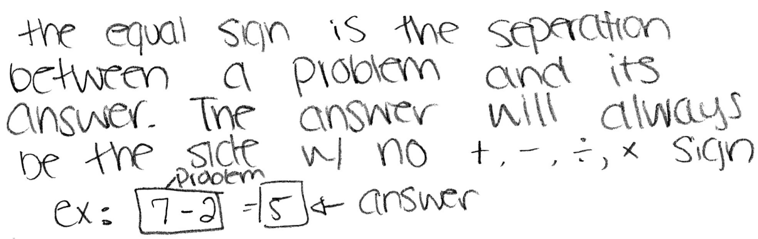

Figure 4: Operational (Kimberly)

There were, however, some relational answers, either strictly so (Felicity: "Something is equal to another thing") or combining operational and relational responses (figure 5).

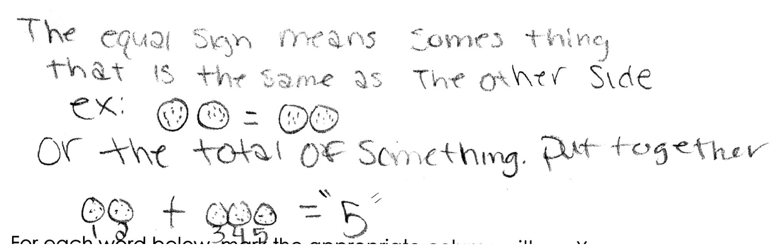

Figure 5: Operational/Relational (Nikki)

Of the twenty-six respondents, eighteen gave an operational response (thirteen rigidly so), three gave a combined response, three gave a solely relational response, and two said they didn't know.

For both of these questions, there was no particular pattern of distribution based on age; in other words, there is no indication among the sample that relational understanding improves during the high school years. Indeed, one of the simplest but most balanced answers to the second question ("The equal sign means the same or total") was provided by 14-year-old Mimi, while four of the 17-year-olds in the sample provided operational answers (Unity: "To put the correct answer or to put the sum."), or the other said she didn't know. Obviously, the total sample size is too small to make definitive generalizations about age regardless.

There was some minor correlation between awareness of the terms (PR-3) and how well the respondents performed on the worksheet, but I did not see any definitive patterns. This question was included primarily in case the open-ended responses were not useful, but the reverse turned out to be the case.

The Nature of Numbers

All respondents correctly answered the Level 1 (NN-1) question, and all but one correctly answered the Level 2 (NN-2) question. The sole incorrect answer to NN-2, Lilac, gave the Level 1 response of 26 (adding 12 and 14 in $12 + \square = 14$); notably, Lilac got all of the other mathematical questions incorrect, but succeeded on three of the five text-based questions.

For NN-3, the non-symmetrical Level 3 question, nine answered correctly with 3, ten answered with 10, four answered with 17, and the remaining three provided a different response (2, $11 + 7 = 18$, and blank). It's possible that Valencia's response of 2 was a result of mathematical error and the thinking was correct; Xandra's $11 + 7 = 18$ could be a mathematical error combined with an attempt to write the entire problem out, splitter style (that is, $8 + 2 = 10 + 7 = 17$).

Overall, then, the respondents did show a movement away from the Level 1 Operational interpretation of the problem discussed in the literature of middle school students. In fact, all seven of the 13 and 14 year old respondents gave an operational response (or left it blank). Discounting Valencia's 2 as a possible error, exactly half of the remaining eighteen 15- to 17-year olds answered this question correctly.

Including Valencia, three of the respondents giving incorrect responses to NN-3 gave correct answers to NN-4, the symmetrical Level 4 question. Most respondents appeared to approach NN-3 and NN-4 the same way: A response of 10 to NN-3 would predict a response of 17 to NN-4, and this was true in six of the ten cases (the others were 5, 5, 15, and 20). A response of 17 to NN-3 would predict a response of 29 to NN-4, and this occurred in two of the four cases (the other responses were 27 and 28, which could both be the result of addition errors). And Xandra applied the same splitter strategy, this time getting it "correct": $12 + 5 = 17 + 12 = 29$.

Overall, then, it does not appear that students approached the symmetrical question NN-4 differently than they approached the non-symmetrical question NN-3. However, there's one interesting tidbit: The two respondents who clearly changed their thinking went from ignoring the 7 entirely to relying on symmetry; those who used all of the numbers and reframed it as a rigidly operational problem did so in both cases. It would be interesting to see if that pattern would hold up on sample sizes of statistically significant size.

Furthermore, NN-4 and NN-5 rely on the same basic knowledge and strategy, but six respondents said NN-5 was true while giving an incorrect answer to NN-4, and one gave the correct answer to NN-4 while saying NN-5 was false.

Correct answers to NN-5 and NN-6 rely on both symmetry and an understanding that addition is commutative while subtraction is not. Eight respondents got both questions correct, while fifteen respondents gave the same answer to both questions (nine said both were true, four said both were false, and two didn't know). NN-7 relies on both symmetry and an understanding of equality above the rigid operational level; fourteen of the twenty-six respondents got this one correct, just above chance. Only six respondents got all three questions correct.

In contrast, all twenty-six respondents gave a correct answer to NN-11, a text-based question relying solely on symmetry ("If Barack Obama is the current President..."), while twenty-one respondents were able to correctly identify this year's quarterback in NN-15. This contrast has two possible interpretations: Students have an easier time with intrinsically mathematical questions as long as they're not presented mathematically, or students have a clear understanding of symmetry in equality but not of issues like commutativity. As I discuss below, I feel the nature of the question (mathematical or non-mathematical) has the greater influence, of these.

NN-8, NN-9, and NN-10 rely solely on transitivity; unlike NN-4 through NN-7, the order of elements within expressions does not change. Students struggled with these questions; eight got NN-8 correct, eight got NN-9 correct, and seven got NN-10 correct. Only three respondents were correct on all three. The most striking example is NN-10 (“If b = 4q...”); NN-12 is a plain text equivalent (“If one dollar is worth...”), which twenty of the twenty-six respondents got correct.

Of the two respondents who were correct on NN-10 but incorrect on NN-12, one answered True to all six of NN-5 to NN-10, suggesting that she may have simply circled T all the way down. This student was Nikki, who admitted to guessing on some of these questions in the one-on-one interview. Intriguingly, though, she also said she particularly didn’t like story problems because her reading is weaker than her mathematics.

A special education teacher who reviewed my survey instruments after the fact commented that NN-10 may be problematic for some students because of the visual similarity of the letters b, d, and q; I’d used those because I’d wanted to have a clear parallel between NN-10 and NN-12, but I do acknowledge the point. Regardless, though, given that students had comparative difficulty between NN-8, NN-9, and NN-10, with the first two using traditional algebraic variable names, I’m not sure this concern affected the data much.

Students did not struggle with the transitive family relationship (NN-13): twenty-two correctly identified Dejuan as Kingsley’s sibling (one said “sister,” perhaps misled by the name Kingsley). They did struggle more with the non-transitive relationship: Twelve correctly identified Esther as Destiny’s grandmother, three said she was the granddaughter, three said she was the mother, four gave another answer, and four didn’t know. Still, their success rate with NN-14 was higher than with NN-10, even though they were asked to provide a written answer rather than simply circling a response.

While there may be math-related mitigating circumstances involving the contrast between the algebraic and text-based symmetry questions, the same don't exist with the transitivity questions, particularly NN-10 and NN-12. In interviews, respondents reported not noticing the similarity between these questions; Bettina commented that NN-12 was easier for her "because I know how to make a dollar" and Nikki said, "I could have done it with actual money."

Group Interviews

Three group interviews were conducted. Group 1 consisted of Amy (who declined the paper questionnaire) and Felicity. Group 2 consisted of Geri, Jackie, and Lilac. Group 3 consisted of Abigail, Bettina, and Nikki. All respondents were from Group A; I did not have the opportunity to interview students from Group B one-on-one.

The questions in the group interview focused on general attitudes toward mathematics, memories about the equality sign and how it impacted their opinion of mathematics, and specific approaches to the questions on the survey.

Students were split as to whether they preferred easy or difficult mathematics. Geri and Lilac (Group 2) said they prefer easy mathematics, while Jackie said, "I feel like easy stuff is taking the easy way out." Amy reported getting frustrated easily, "I don't like numbers"; Felicity said she used to hate mathematics when she was younger (she'd put peanut butter on her homework and feed it to her dog), but equations make it fun, and while she feels that math doesn't come easily for her, "I like to do really hard equations, I like to solve them." Nikki said, "I like it until I don't understand it"; in particular, she doesn't like anything involving reading (although her performance on the questionnaire doesn't match this self-assessment).

Most interviewees didn't clearly remember their early encounters with the equal sign; Abigail said she's always been good at math ("My very first math test, I got 100%"). Those who did remember had an operational memory:

FELICITY: When I was young, two plus two equals, and then...

AMY: It was hard. I thought it was hard.

FELICITY: I just thought it was hard that the two numbers added up and on the other side was the answer.

and:

JACKIE: My daddy taught me before school that's the total of whatever.

LILAC: That's the sum of a number.

As for math class, interviewees reported trying to avoid it. Bettina said she does better with tutors, while Felicity said, "I hated the homework and the class. I used to sleep; the teacher would throw an eraser at my head." Geri said she doesn't like math class when she doesn't know how to do it, but likes the one-on-one treatment she gets at the current school.

When discussing NN-3 and NN-4, students generally gave relational explanations, although some struggled with articulating them:

HARTZER: Nikki, how did you approach 3?

NIKKI: I don't know.

HARTZER: Abigail, how about you?

ABIGAIL: I just saw that one side equals ten. So I said seven plus what equals ten.

NIKKI: Yeah, that's what I did.

Felicity explained, "Eight plus two is seven plus what, they have to be equal to each other." However, Amy solved them operationally during the interview: "Eight plus two is ten plus seven is seventeen? Twelve plus five is seventeen plus twelve is twenty-four?" She did it this way despite having just heard Felicity providing a different solution and explanation.

While they couldn't explain why, several students did say that problem NN-4 was easier than NN-3.

When discussing NN-5 and NN-6, Felicity noticed her error in NN-6: "Six minus three equals three and three minus six is oh... I got that wrong. I was rushing." Nikki also suggested she wasn't paying enough attention, which lead Abigail to tease her:

BETTINA: When you subtract a big number from a little one, you get a negative.

NIKKI: [The two expressions] look the same so I just put true.

ABIGAIL: She's a bubblehead.

NIKKI: I don't see the difference. They look the same on each side.

ABIGAIL: Three minus six is negative three.

NIKKI: Oh, I see.

I was particularly interested in the difference between NN-10 and NN-12, given that of one of my research questions concerned whether difficulties with the relational view extend to non-mathematical contexts. None of the students interviewed had noticed the similarity in the questions when completing the worksheet, and all but one of them said that question NN-12 was easier. Jackie was the exception: "I think about them the same. Ten just came to me. I don't know." Lilac responded, "Twelve is easier because I can read the problem without the numbers" and Geri commented that "Twelve was a lot easier." Similarly, Felicity said, "I didn't notice they were similar. I didn't have trouble when I read twelve. It's easiest because it's written out." Group 3 agreed:

BETTINA: Twelve is easier because I know how to make a dollar.

ABIGAIL: I already know that ten dimes makes a dollar, but it's like in geometry.

NIKKI: Because, I don't know... I could have done it with actual money.

One important element in these reports is the use of prior knowledge. After the questionnaire, Rachel said she couldn't do NN-15 because, "I don't know anything about the Wildcats." While most students provided the correct answer for NN-15, *every* student provided the correct answer for NN-11, supporting the view that prior knowledge can be helpful in solving problems.

Conclusions and Implications

The tradition of behaviorism-based education has led to a preponderance of what Skemp (2006) terms "instrumental understanding," that is, teaching formulas and procedures rather than teaching core mathematical concepts ("relational understanding"). These two types of understanding are reflected in the basic perspectives about equality. An operational understanding comes from instrumental thinking, that is, that the point and purpose of mathematics is to solve problems that are presented, and that therefore there ought to be a specific place for the problem (generally, the left side) and a specific place for the solution (generally, the right side). A relational understanding, however, reinforces that the primary point of the equality sign is to indicate that its two sides represent equivalent values; any problem that's presented consists of making adjustments or finding values that would make the overall statement true.

This relational understanding is key to success in mathematics beyond and even including simple arithmetic; as Jones et al. write, "A sophisticated and flexible understanding of the equals sign (=) is important for arithmetic competence and for learning further mathematics, particularly algebra" (Jones, Inglis, Gilmore, & Dowens, 2012, 166). Jones et al. characterize two components of a relational conception: That, if $a = b$, then a and b represent the same value ("sameness") and that a can be replaced with b without affecting the total value of an expression ("substitution").

The basic concept of symmetry relies on sameness, while the basic concept of transitivity relies on substitution. Hence, a poor understanding of those concepts will negatively impact an overall quality understanding of the function of the equality sign, particularly with regards to the relational view.

As Skemp (2006) and Prediger (2010) discuss, it is key for an effective teacher to be able to understand how students are seeing and approaching problems. If a teacher is taking a relational approach and assuming that students are doing the same when approaching a problem such as, for instance, $8 + 5 = \square + 7$, then this will create a counterproductive wall to understanding.

This research was designed to look at two key questions:

1. Do difficulties with the relational view extend to transitivity and symmetry?
2. Do difficulties with the relational view extend to non-mathematical contexts?

The answer to the first question appears to be, "Yes." While students performed slightly better on mathematical problems based on symmetry and transitivity than on others, and felt that such problems were a little easier, they still struggled. Those students who are still at the middle-school level of focusing primarily on operational perspectives performed only slightly better on the symmetry and transitivity problems.

One piece of good news in this data is that the transition towards relational thinking that Wheeler (2010) worries about at the college level has clearly already begun at the high school level. This does somewhat support Kieran's view (1981) that there are physiology-based cognitive limitations in primary-age students which prevent understanding, but this shift could also be attributed to a concentrated shift during high school from arithmetic to algebra. For secondary teachers, one possible implication of this is that more emphasis should be placed in exercises and presentation on establishing a relational view of equality.

However, there was a disconnect between the worksheet data and the reflection data. While students are revealing a higher level of relational thinking when it comes to actually solving problems, their articulations of operational vs. relational views remain fairly constant throughout high school. As Marcial and Marcial (Marcial & Marcial, 2004, 8) warn, "Just because students may perform well on a mathematics assessment does not necessarily mean that they understand what they are doing."

The answer to the second question appears to be a cautious, "No." The data itself supports the notion that mathematical problems situated in non-mathematical frames are approached differently. This is supported not just by the battery of questions NN-11 to NN-15, but also by the responses to PR-1, where most students offered some sort of "real world" explanation of the problem, either through actual objects (apples, hearts, cookies, pickles, or pencils) or through tally marks or finger counting. Only a third stayed entirely in the mathematical abstract.

However, one mitigating element is that of prior knowledge. When students are able to fully visualize a problem because it ties to what they already know, the mathematical element becomes less challenging. In future research, it would be interesting to explore the impact of replacing the questions that rely on such prior knowledge with others (an early draft of my instrument, for instance, included a question along the lines of, "If there are three griffs in a dorn and five dorns in a whimp, how many griffs are in a whimp?"). It is true that students performed fairly well on a non-mathematical symmetry problem that did not rely on prior knowledge (NN-15), but they did much better on the one that did (NN-11).

The point of the caution is that the response to this data is not to simply convert to story problems entirely. In my other explorations with story problems, students do struggle with parsing the problem if it's too complicated or remote to their knowledge. This data and that experience indicate that, if a student can parse the sense of a story problem, they can solve the problem successfully, even if they can't articulate it in an algebraically meaningful way. For instance, on a story problem exercise, students did fine with a question along the lines of, "Mom sends Julie to the store to buy three loaves of bread. Mom gives her $20. If the bread is $2.95 a loaf, how much change does Julie bring home?" The students knew to take 20 and subtract 2.95 three times, but they didn't know how to write that as a formula. In another case, students in Group B struggled with the notion of *depreciation* when it applied to buying a car (remote to their knowledge), but understand very well when I used an example of buying a new television (within their knowledge base) instead.

Overall, if a student cannot parse a story problem, this adds another level of frustration: Not being able to convert the problem into something mathematically meaningful (algebraic or otherwise), and then not being able to solve the problem.

Furthermore, math anxiety and a feeling of hopelessness is common in students, particularly by the time they get to the secondary level. During interviews, those students who struggled with the questionnaire reported guessing or rushing at several points. Bettina said, "We feel like we're guessing a lot in math class. It's a common feeling."

On the other hand, the most obvious way to make mathematics meaningful is to present it in a context that has real-world meaning. While students complain about "story problems," they also appear to make a distinction between real-world, practical problems (such as the aforementioned loaves of bread question) and story problems that have been contrived in order to demonstrate or even hide a mathematical lesson (e.g., "Jim's Dad is three times his age. In ten years, he will be twice his age. How old is Jim now?").

The overarching implication of the data matches that of the literature on the original subject. While students at the secondary level show an emerging understanding of the relational view of the equality sign, it is important that teachers (1) keep in mind that this is an emerging rather than perfected understanding and (2) strive to find ways to reinforce this understanding by presenting non-operational contexts in which to explore equality (Oksuz, n.d.).

Appendices

Appendix A: Survey Instruments

Equality: Preview Reflection

Your FIRST NAME: _________________ Your AGE: ______ Your GRADE: ______

Please think about the following questions and provide your thoughts on them. You are not being graded. The purpose of this exercise is to get an understanding of what you already know so that we can improve education for you and other students in the future. **Only Mr. Hartzer will know your name; it will not be shared with ANYONE else.**

1. Describe how you would explain the following problem: 3 + 5 = _____

2. Think about the equal sign (=). What does it mean?

3. For each word below, mark the appropriate column with an X:

	I can explain this MATH TERM	I've heard of this MATH TERM, but can't explain it	I've heard this term, but not in MATH class	I've never heard of this term before
Transitive				
Symmetry				
Commutative				
Associative				

The Nature of Numbers

Your FIRST NAME: ________________ **Your AGE:** ______ **Your GRADE:** ______

Please complete the following questions the best that you can. You are not being graded. If you don't know the answer to a question, do your best to answer it.

Write the number that goes in each box:

1) 3 + 1 = ☐
2) 12 + ☐ = 14
3) 8 + 2 = ☐ + 7
4) 12 + 5 = ☐ + 12

ANSWER
1)
2)
3)
4)

Indicate whether these statements are true (T) or false (F), or you can't tell (DK):

5) 4 + 3 = 3 + 4
6) 6 – 3 = 3 – 6
7) If 5 + x = 8 then 5 = 8 + x
8) If 2 + x = 1 + y and 1 + y = 5 then 2 + x = 5
9) If x = 2 + y and 2 + y = z then x = 2 + z
10) If b = 4q and 4q = 10d then b = 10d

5) T F DK
6) T F DK
7) T F DK
8) T F DK
9) T F DK
10) T F DK

Answer these questions YES or NO:

11) If Barack Obama is the current President of the United States, can we say for sure that the current President of the United States is Barack Obama?
12) If one dollar is worth the same as four quarters and four quarters is the same as ten dimes, can we say for sure that one dollar is worth the same as ten dimes?

11) Yes No
12) Yes No

Fill in the blank:

13) Dejuan is Louis's brother. Louis is Kingsley's brother. Therefore, Dejuan is Kingsley's ___________.
14) Esther is Janelle's mother. Janelle is Destiny's mother. Therefore, Esther is Destiny's ___________.
15) Lebron Willis is this year's quarterback for the Washington Wildcats. This year's quarterback for the Washington Wildcats is _________________.

13)
14)
15)

Equality Sign Research Project

As a group, have the students complete the “preview reflection” and the “nature of numbers” worksheets. Select individuals for further interviewing, as below.

1. In general, how do you feel about mathematics now? Can you describe the sorts of feelings you have when you think about doing a mathematics problem? How do you feel when you think about going to mathematics class?

2. Think back on one of the first times you can remember a teacher discussing the equal sign. It doesn’t have to be the first time you ever encountered it, just an early time that you remember. Do you remember when it was? What did you think of it? At the time, how did you feel about mathematics? *(If attitudes have changed, ask:)* Why do you think you feel differently about mathematics now?

3. For each of the following questions on the “nature of numbers” sheet, ask: Tell me about question ____. Can you explain to me how you got that answer?

   Repeat for questions 3, 4, 5 vs 6, 7*, 8*, 9, 10 vs 12, 11*, 13 vs 14*, 15. *Esp. if wrong

Appendix B: Survey Instrument reference/answers

Equality: Preview Reflection

Key	Question
PR-1	Describe how you would explain the following problem: $3+5=$ ____
PR-2	Think about the equal sign (=). What does it mean?
PR-3	For each word below, mark the appropriate column.
PR-3a	Transitive
PR-3b	Symmetry
PR-3c	Commutative
PR-3d	Associative

The Nature of Numbers

Key	Question	Answer	Property
NN-1	$3+1=\square$	4	Level 1
NN-2	$12+\square=14$	2	Level 2
NN-3	$8+2=\square+7$	1	Level 3
NN-4	$12+5=\square+12$	5	Symmetry
NN-5	$4+3=3+4$	True	Symmetry
NN-6	$63=36$	False	Symmetry
NN-7	If $5+x=8$ ...	False	Symmetry
NN-8	If $2+x=1+y$ ...	False	Transitivity
NN-9	If $x=2+y$ ...	True	Transitivity
NN-10	If $b=4q$ and ...	True	Transitivity
NN-11	If Barack Obama is the current ...?	Yes	Symmetry
NN-12	If one dollar is worth ...?	Yes	Transitivity
NN-13	Dejuan is Louis's brother....	Brother	Transitivity
NN-14	Esther is Janelle's mother....	Grandma	Transitivity
NN-15	Lebron Willis is this year's ...	Lebron	Symmetry

Appendix C: Initial Survey Design

Name(s)____ Date__________________________

Project Design and Data Catalog

PROJECT DESIGN

State your research question(s).

The equality sign in mathematics indicates that the values represented by the expressions to either side of the sign are equivalent. An understanding of this notion is crucial to success in algebra. However, several studies (e.g., Oksuz 2001; Rittle-Johnson, Matthews, Taylor, McEldoon 2010) have shown that students entering high school are more inclined to think of the equality sign as an operator: In “3 + 5 = 8”, for instance, the equality sign is seen as a trigger to add three and five (to get eight). There has been extensive study of questions of the form “3 + 5 = [ ] + 2”, where students are generally inclined to insert eight rather than six in the blank. This study seeks to explore transitivity (if a = b and b = c then a = c) and symmetry (if a = b then b = a); the initial theory is that students who see the equality sign as a trigger for performing a calculation will likewise struggle with these concepts, even though they’re foundational in arithmetic (Kieran 1981).
The proposed study will rely on three basic instruments. I will begin by having students complete a three-question “Preview Reflection” on paper. Then I will have them complete a brief written problem set related to mathematical understanding of equality. From this pool of students, I will interview three or four students of varying levels of skill.

DATA CATALOG

Please complete the following two tables about your data collection and analysis plans. Staple the most recent drafts of your instruments and any coding keys to this document.

Name(s)____ Date____________________

Data Source (include number of participants)	What you will learn from the data source?	Date(s) for data collection	Note any special circumstances surrounding data collection
1. Preview reflection on equality (10-15 HS students)	• I will get an idea of basic understanding of equality prior to more focused questions. I will also test awareness of key mathematical terms.	A single date in June	
2. Written assessment on equality, transitivity, and symmetry (10-15 HS students)	• I will identify current understanding of transitivity and symmetry in high school students. • Based on the results of this assessment, I will select 3 or 4 students for one-on-one interviews. I will select a mix of low, medium, and high performing students.	A single date in June (immediately following reflection)	
3. One-on-one interviews (3-4 students)	• I will probe deeper on specific topics related to student understanding of the equality sign. I will gain a better understanding of how student understanding of equality shifts as they transition from arithmetic to algebraic thinking.	Several days in June	

Name(s)____ Date___________________________

Data Source	Initial Plans for Data Analysis
1. Reflective writing	• Code data based on specific concepts mentioned • Include a comparison in the coding to allow for pre vs post knowledge
2. Assessment	• Code data based on correct responses • Select 4-6 students as potentials for interviewing one-on-one based on performance (low, med, high)
3. Interviewing	• Code data based on specific concepts mentioned • Interview notes will be tied to assessments, but otherwise no identifying information will be recorded; only pseudonyms will be used in any notes • Note: These interviews cannot be recorded; coding will be based on interviewer's notes

Appendix D: Respondent List

Pseudonym	Group	Race	Age	Grade	Interview
Abigail	A	White	17	12	3
Amy	A	Black		10	1*
Bettina	A	White	15	9	3
Chantel	A	Black	16	10	
Daisy	A	Black	17	10	
Ebony	A	Black	15	10	
Felicity	A	White	16	11	1
Geri	A	Black	16	11	2
Helen	A	Black	17	12	
Isabella	A	Black	15	10	
Jackie	A	Black	15	9	2
Kimberly	A	Black	16	10	
Lilac	A	Black	16	10	2
Mimi	A	Black	14	9	
Nikki	A	White	16	9	3
Olivia	B	Black	14	9	
Pamela	B	Black	13	7	
Quinn	B	Black	13	7	
Rachel	B	Black	13	7	
Samantha	B	Black	17	11	
Twilight	B	Black	14	9	
Unity	B	Black	17	11	
Valencia	B	White	16	10	
Wendy	B	Black	13	9	
Xandra	B	Black	15	9	
Yvonne	B	Black	16	10	
Zinnia	B	Black	15	10	

* Interview only (refused questionnaire)

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